**Question 3:**

The range [1, n] is a discrete uniform distribution, so we firstly set out to recreate a uniform distribution of unique random numbers

**Creating Uniform Random Number Distributions**

We used a MCG (Multiplicative Congruential Generator) to generate (pseudo) random numbers, with

a = 16807,

m = 2^31 – 1, and

Seed = (Java-generated random number in the range [1, 57689])

We chose the range [1, 57689], because the seed cannot equal 0 ( so the lower bound was 1), and we chose 57689 (the upper bound) randomly. That is, we had a different seed for every simulation

**Verifying Uniformity**

Since the period (without repetition) is m - 1 (~ 2.15 \* 10^9), and we generated distributions with a maximum length of (1 \* 10^3).

(~2.15 \* 10^9) > (1 \* 10^3) ==

(m-1) > (n)

The period was larger than the sample size. Thus, we know that the numbers were distributed uniformly, and every number in the set was unique.

**Chi-Square**

Moreover, the results of our chi-square tests, were less than 79.96 for the corresponding value on the chi-square distribution table for n-1 degrees of freedom and α = 0.05

n = number of elements in distribution

d.f = 99, α = 0.05, x^2 = ?????????

d.f = 999, α = 0.05, x^2 = ?????????

Therefore, due to the period of the MCG being larger than the sample size, and scores that validate against the Chi-Square test, we believe that our random number distributions were uniform

**Question 3 Introduction:**

In this question, we have developed 3 strategies to determine the highest number among n positive numbers, to win €1000. Strategy 1 is designed for someone who has no data on previous outcomes of the game. Strategy 2 demonstrates the value, or rather, lack of value, in random selection. Strategy 3 is intended for a player who has data on the past outcomes of the game’s random number generator.

**Strategy 1 ‘Point Of Desperation’:**

Our first strategy is to conditionally select our number past a ‘point of desperation’ (POD).

In this strategy, we define a ‘point of desperation’ (POD). As we move through the uniform distribution of unique numbers, we record, but don’t select, the highest number we encounter before the POD – our ‘record highest’. Then, once we pass this POD, we select the first number that is higher than our ‘record highest’

So, if our POD is n/4, we record, but don’t select, the highest number from [1, n/4]. Then, we select the first number from [(n/4) +1, n] that is higher than our ‘record highest’.

We will test this with distributions of n = 10, 100, 1000 (with 1 000, 10 000, 100 000 simulations each) at PODs n/4, 2n/4, 3n/4

These are the raw results;

A table with numbers and points

Description automatically generated

Bar-chart:

We can see that fn(A) (relative frequency of success) is highest for a POD of n/2 (with distributions of n = 1000)

P(n/2) = 0.25

**Strategy 2 Conclusion**

Therefore, the player should choose a POD of n/2 (or exact middle value) to maximise their chances of winning with this strategy

**Strategy 2 ‘Random Selection’:**

Our second strategy is to choose a random number/piece of paper. While it is a rudimentary strategy, with a probability of success of 1/n, we feel it will prove the advantages of other strategies.

These are the raw results:

A table with numbers and a number of objects

Description automatically generated with medium confidence

And these are the relative frequencies of success with 100,000 trials for n = 10, 100, 1000 in that order

We can see that there is a negative linear relationship with the size of n and the relative frequency of winning – as n increases, the player’s chances of winning decrease

**Strategy 2 Conclusion**

Thus, we advise to pursue strategy 1 over strategy 2 when n > 4

**Strategy 3 ‘Highest Range’:**

Our third strategy is inspired by the counting-cards strategy in Blackjack. It is intended for real-life scenarios, where random number generators are deterministic, and thus have deterministic upper bounds.

In this strategy, we determine find the range which the highest numbers are found, prove that RNGs have a determined range of upper bounds, through Monte Carlo simulations, which will give us a discrete random variable to develop a flexible strategy.

To find this range, we recorded the highest number in each of 1,000, 10,000 and 100,000 distributions/simulations (with n = 1000 , based off of the results of strategy 1), and plotted them into discrete random variables. In these distributions of highest numbers, we found the smallest and largest number, the expected value and the variance.

**Smallest** overall number that was highest in its distribution = **Highest Floor**

**Largest** overall number that was highest in its distribution = **Highest Ceiling**

Highest range = Highest Floor - Highest Ceiling

A screenshot of a white sheet

Description automatically generated

* One observation is that as you increase n by 10^x, the variance in highest numbers obtained decreases by 10^x
* This satisfies the empirical law of large numbers
* Therefore, it is in the player's best interest to play with a **larger** n (assuming that they are patient enough) in this strategy

The larger our sample size, the narrower our guess range of numbers, meaning we can more accurately guess the highest number.

With this information, we advise to play games with the highest n, to further your chances of winning

Now, we will run a modified version of strategy 1. We will define a ‘point of desperation’ (POD). A POD is, if we pass this point, and have not achieved a desirable outcome so far, we will select the next number in the range [highest\_floor, highest\_ceiling], that is higher than every number we’ve already come across so far.

We will test this strategy with PODs n/4, 2n/4, 3n/4 (ie, Quartile 1, Median and Quartile 3)

A desirable outcome is related to the POD. If the POD is n/4, the range for a desirable outcome increases by 4% (of the total distribution of highest numbers) with every increment towards n.

So, if POD = n/4, and we have taken 2 steps, we will select any number that appears in the range, [ (highest\_ceiling – (highest\_range\*(2\*(4/n)) ) ), highest\_ceiling]

k = number of steps taken

c = inverted POD (ie, if POD = n/2, c = (2/n),

if POD = 2n/4, c = (4/2n),

if POD = 3n/4, c = (4/3n)

Range of desirable outcome = [ (highest\_ceiling – (highest\_range\*(kc) ) ), highest\_ceiling]

This is based on the strategy of counting cards in Blackjack, where if you get a high number and not many cards have been dealt, you stick with that number. However, if the dealer has dealt many cards and your hand is still bad, you’re more likely to take a risk. Similarly, we are more likely to take a risk with a *relatively* low high number.

Finding the highest ceilings, floors and range, with different parameters to the MCG than in the first picture:

A white sheet with black text

Description automatically generated

Here, you can see that values have large differences in magnitude to the earlier first picture, despite them both being random number generators. This proves that the MCG, while providing a uniform distribution, has an upper limit, regardless of the seed.

Since it is not a truly random distribution, our strategy to determine the lower and upper limits of the game’s random number generator is valid. Here are the results, when we know the range of upper limits of the RNG:

A white grid with black text

Description automatically generated

We can see that Strategy 3 is more successful than the previous two strategies.

However, it requires a lot of prior data on the previous outcomes of the random number generator, and hinges on the probability that the game owners do not change their RNG

Thus, we conclude that it is in the players’ best interests to player with a larger *n*, and a POD of 9n/10

**Question 3 Conclusion:**

Strategy 2, ‘Random Selection’, scales very poorly, so do not choose Strategy 2 (unless n < 5, which is extremely small, and it is unlikely that the game owners will choose a value of *n* that small). This has a probability of success of 1/n – the lowest value possible

Strategy 1, ‘Point Of Desperation’, with a POD of n/2, is the best strategy for a player who has no prior data on the results. This has a probability of success of 0.25 at best (making it better than Strategy 2 for n > 4)

Strategy 3, ‘Highest Range’ has the highest probability of success of any strategy, however, it based on practical assumptions, rather than theoretical foundations.

We conclude that the theoretically best strategy is Strategy 1, ‘Point Of Desperation’, however, if certain conditions are met (data collected on > 1,000 distributions of size 1000, the game owners have used and will use the same deterministic random number generator), the practically best strategy is Strategy 3, ‘Highest Range’

**Code: ®**